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Parametrization by fixed-points multipliers of the polynomials with degree n

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1 Introduction

Let $\mathrm{Poly}_n(\mathbb{C})$ be the the polynomials from the Riemann sphere, $\widehat{\mathbb{C}}$, to itself, with degree n, and $\mathbb{M}_n$, called moduli space, the quotient space of $\mathrm{Poly}_n(\mathbb{C})$ under the action of the affine transformation group,$\mathfrak{A}(\mathbb{C})$.

We parametrize $\mathbb{M}_n$ by using multipliers of fixed points, and define a natural map Ψ from $\mathbb{M}_n$ to $\mathbb{C}^{n-1}$. A new coordinate system is called multiplier cooridinates. Exhibiting the moduli space of a higher degree under this system deserves particular attention. For example, in study of geometry and topology of $\mathrm{Poly}_n(\mathbb{C})$ from a viewpoint of complex dynamical systems, we make use of this system in order to express singular part, and dynamical loci as algebraic curves or surfaces([NF99], [NF00]).

The subject of this paper is surjectivity-problem of the map Ψ from $\mathbb{M}_n$ to $\mathbb{C}^{n-1}$: a problem of characterization of exceptinal part, $\mathcal{E}_n(=\mathbb{C}^{n-1}\setminus\mathbb{M}_n)$.

The initiator of the use of multiplier cooridinates is J. Milnor ([Mil93]), to the case of the quadratic rational maps.

2 Polynomials of degree n

2.1 Polynomial maps and conjugacy

Let $\widehat{\mathbb{C}}$ be the Riemann sphere, and $\mathrm{Poly}_n(\mathbb{C})$ be the space of all polynomial maps of degree n from $\widehat{\mathbb{C}}$ to itself:

$$p(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0 \quad (a_n\neq 0).$$

The group $\mathfrak{A}(\mathbb{C})$ of all affine transformations acts on $\mathrm{Poly}_n(\mathbb{C})$ by conjugation:

$$g\circ p\circ g^{-1}\in \mathrm{Poly}_n(\mathbb{C}) \quad \text{for} \quad g\in\mathfrak{A}(\mathbb{C}),\ p\in\mathrm{Poly}_n(\mathbb{C}).$$

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Two maps $p_1, p_2 \in \mathrm{Poly}_n(\mathbb{C})$ are **holomorphically conjugate** if and only if there exists $g \in \mathfrak{A}(\mathbb{C})$ with $g \circ p_1 \circ g^{-1} = p_2$.

Under this conjugacy of the action of $\mathfrak{A}(\mathbb{C})$, any map in $\mathrm{Poly}_n(\mathbb{C})$ is conjugate to a "monic" and "centered" map, i.e.,

$$p(z) = z^n + c_{n-2}z^{n-2} + c_{n-3}z^{n-3} \cdots + c_0.$$

We remark that this p is determined up to the action of the group $G(n-1)$ of $(n-1)$-st roots of unity, where each $\eta \in G(n-1)$ acts on $p \in \mathrm{Poly}_n(\mathbb{C})$ by the transformation $p(z) \mapsto p(\eta z)/\eta$.

Every polynomial map from $\widehat{\mathbb{C}}$ to itself is conjugate under an affine change of variable to a monic centered one, and this is uniquely determined up to conjugacy under the action of the group $G(n-1)$ of $(n-1)$-st roots of unity.

For example,in the case of $n = 3$, the following two monic and centered polynomials belong to the same conjugacy class:

$$z^3 + az + +c, \; z^3 + az - c.$$

In the case of $n = 4$ the following three monic and centered polynomials belong to the same conjugacy class:

$$\begin{aligned} &z^4 + az^2 + bz + c \\ &z^4 + a\omega z^2 + bz + c\omega^2 \\ &z^4 + a\omega^2 z^2 + bz + c\omega \end{aligned}$$

where ω is a third root of unity.

2.2 Moduli space of polynomial maps

The quotient space of $\mathrm{Poly}_n(\mathbb{C})$ under the action $\mathfrak{A}(\mathbb{C})$ will be denoted by $\mathbb{M}_n$, and called the **moduli space** of holomorphic conjugacy classes $\langle p \rangle$ of polynomial maps p of degree n.

Let $\mathcal{P}_1(n)$ be the affine space of all monic centered polynomials of degree n

$$p(z) = z^n + c_{n-2}z^{n-2} + c_{n-3}z^{n-3} \cdots + c_0,$$

with coefficients-coordinate $(c_0, c_1, \cdots, c_{n-2})$.

Then we have an $(n-1)$-to-one canonical projection Φ from $\mathcal{P}_1(n)$ onto $\mathbb{M}_n$.

Hence the affine space $\mathcal{P}_1(n)$ is regarded as an $(n-1)$-sheeted covering space of $\mathbb{M}_n$. Thus we can use $\mathcal{P}_1(n)$ as a coordinate space for the moduli space $\mathbb{M}_n$, though it remains the ambiguity up to the group $G(n-1)$. This coordinate space has the advantages of being easy to be treated.

However, it would be also worthwhile to introduce another coordinate system having any merit different from $\mathcal{P}_1(n)$'s.

In fact, Milnor successfully introduced coordinates in the moduli space of the space of all quadratic rational maps using the elementary symmetric functions of the multipliers at the fixed points of a map ([Mil93]). To the case of $\mathrm{Poly}_n(\mathbb{C})$, we try to explore an analogy.

2.3 Multiplier coordinates

Now we intend to explore another coordinate space for $\mathbb{M}_n$. For each $p(z) \in \mathrm{Poly}_n(\mathbb{C})$, let $z_1, \cdots, z_n, z_{n+1}(=\infty)$ be the fixed points of p and μ_i the multipliers of z_i ; $\mu_i = p'(z_i)$ $(1 \leq$

$i \le n)$, and $\mu_{n+1} = 0$. Consider the elementary symmetric functions of the n multipliers,

$$\begin{aligned}
\sigma_{n,1} &= \mu_1 + \cdots + \mu_n, \\
\sigma_{n,2} &= \mu_1\mu_2 + \cdots + \mu_{n-1}\mu_n = \textstyle\sum_{i=1}^{n-1} \mu_i \sum_{j>i}^{n} \mu_j, \\
&\cdots \\
\sigma_{n,n} &= \mu_1\mu_2\cdots\mu_n, \\
\sigma_{n,n+1} &= 0.
\end{aligned}$$

Note that these are well defined on the moduli space $\mathbb{M}_n$, since μ_i's are invariant by affine conjugacy.

2.3.1 The holomorphic index fixed point formula

For an isolated fixed point $f(x_0) = x_0, \quad x_0 \ne \infty$ we define the holomorphic index of f at x_0 to be the residue

$$\iota(f,\ x_0) = \frac{1}{2\pi i}\oint \frac{1}{z - f(z)} dz$$

For the point at infinity, we define the residue of f at ∞ to be equal to the residue of $\phi \circ f \circ \phi$ at origin, where $\phi(z) = \frac{1}{z}$. The Fatou index theorem (see [Mil90]) is as follows:
For any rational map $f\ :\ \mathbf{C} \to \mathbf{C}$ with $f(z)$ not identically equal to z, we have the relation $\sum_{f(z)=z} \iota(f,\ z) = 1$. This theorem can be applied to these μ_i's ; $\sum_{i=1}^{n} \frac{1}{1-\mu_i} + \frac{1}{1-0} = 1$, provided $\mu_i \ne 1 (1 < i < n)$. Arranging this equation for the form of elementary symmetric functions, we have

$$\gamma_0 + \gamma_1\sigma_{n,1} + \gamma_2\sigma_{n,2} + \cdots + \gamma_{n-1}\sigma_{n,n-1} = 0$$

where

$$\gamma_k = (-1)^k n \binom{n-1}{k} \Big/ \binom{n}{k} = (-1)^k(n-k).$$

Note that $\mu_i = 1\ (1 \le i \le n)$ is allowable here. Then we have the following Linear Relation : •
For the cubic case ($n = 3$),we have $3 - 2\sigma_{3,1} + \sigma_{3,2} = 0$
• For the quartic case ($n = 4$), we have $4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0$
And in general the following linear relation holds:

Theorem 1 *Among $\sigma_{n,i}$'s, there is a linear relation*

$$\sum_{k=0}^{n-1} (-1)^k (n-k)\sigma_{n,k} = 0, \tag{1}$$

where we put $\sigma_{n,0} = 1$.

In view of Theorem 1, we have the natural map Ψ from $\mathbb{M}_n$ to $\mathbb{C}^{n-1}$ corresponding to

$$\Psi(< p >) = (\sigma_{n,1}, \sigma_{n,2}, \cdots, \sigma_{n,n-2}, \sigma_{n,n}).$$

We remark that $\Psi(\mathbb{M}_n) \subset \mathbb{C}^{n-1}$.

2.3.2 Characterization of exceptional set

To investigate whether this map Ψ is surjective or not is our main subject: a problem of characterization of the part of $\mathbb{C}^{n-1} \setminus \Psi(\mathbb{M}_n)$.

We call this set **exceptional set** and denote it by

$$\mathcal{E}_n = \mathbb{C}^{n-1} \setminus \Psi(\mathbb{M}_n).$$

Our main subject is as follows:
For a given $(s_1, s_2, \cdots, s_{n-2}, s_n) \in \mathbb{C}^{n-1}$, we set s_{n-1} a solution of

$$\Sigma_{k=0}^{n-1}(-1)^k(n-k)s_k = 0, \quad s_0 = 1.$$

Then for the point $(s_1, \cdots, s_n) \in \mathbb{C}^{n-1}$, we set a polynomial

$$m(z) = z^n + s_1 z^{n-1} + s_2 z^{n-2} + \cdots + s_{n-1}z + s_n$$

Then w denote the roots of this polynomial by

$$\mu_1, \mu_2, \cdots, \mu_{n-1}, \mu_n.$$

Can we obtain a polynomial $p(z) \in \mathcal{P}_1(n)$ **whose multiplier-coordinate** $(\sigma_1, \cdots, \sigma_n)$ **is corresponding to** $(s_1, \cdots, s_n)$ **?**

Namely can we find a polynomial satisfying that for fixed points z_i

$$p(z_i) = z_i, \quad (i = 1, \cdots, n) \text{ with } \mu_i = p'(z_i).$$

The case $n = 3$ is nicely solved: Ψ is surjective. ([NF96], [FN97]. This fact is mentioned in [Mil93] without any details.)

We also solved this problem for the case $n = 4$ ([NF96], [FN97]):

Theorem 2 $\Psi : \mathbb{M}_4 \longrightarrow \mathbb{C}^3$ *is not surjective:*

$$\begin{aligned} \mathcal{E}_4 &= \mathbb{C}^3 \setminus \Psi(\mathbb{M}_4) \\ &= (4, s, \frac{s^2}{4} - 2s + 4) \;\; s \neq 4 \end{aligned}$$

As for the cases of general n, we expect analogous results.

Recently, we have a following result:

Theorem 3 *(M.FUJIMURA)*
Let $\Omega = \{\mu_i\}_{i=1,\cdots,n}$ be the set of all roots of a polynomial $m(z)$. If Ω satisfies one of the following cases (A),(B) and(C), then there exists a polynomial $p(z) \in \mathcal{P}_1(n)$ such that

$$p(z_i) = z_i, \quad (i = 1, \cdots, n) \textit{ with } \mu_i = p'(z_i).$$

(A):

1. Any element of Ω is not equal 1 :$\mu_i \neq 1$,

2. $\sum_i \frac{1}{b_i} = 0,\; b_i = 1 - \mu_i$,

3. for any proper subset ω of roots, $\sum_{s\in\omega}\frac{1}{b_s}\neq 0$,
(B):
1. Let $\Omega' = \{\mu_i\}_{i=1,\cdots,m}$ $1\leq m\leq n-2$ *be a subset of* Ω *whose elements are not equal 1* $:\mu_i\neq 1$,
2. for any subset ω of Ω' , $\sum_{s\in\omega}\frac{1}{b_s}\neq 0$,
(C):
1. Any element of Ω *is equal 1* $:\mu_i = 1$.

2.3.3 Examples

We shall show some examples for ous inverse problem. By these examples show that the Fujimura's theorem only gives a sufficient condition for surjectivity.

- For a set $\{\mu, 2-\mu, \lambda, 2-\lambda\}$; $\mu\neq\lambda$, $\mu\neq 1$ a corresponding polynomial exits in $\mathcal{P}_1(4)$.
- For a set $\{\mu, 2-\mu, \mu, 2-\mu\}$ $\mu\neq 1$, no corresponding polynomial exits $\mathcal{P}_1(4)$.
- For a set $\{\mu, \mu, \mu, \lambda, \lambda\}$, $\mu\neq 1$, $5-2\mu-3\lambda=0$ a corresponding polynomial exits $\mathcal{P}_1(5)$.
- For a set $\{\mu, \mu, \mu, 2-\mu, \frac{3-\mu}{2}\}$, $\mu\neq 1$, no corresponding polynomial exits $\mathcal{P}_1(5)$.

References

[FN97] M. Fujimura and K. Nishizawa. Moduli spaces and symmetry loci of polynomial maps. In W. Küchlin, editor, *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation*, pages 342–348. ACM, 1997.

[Mil90] J. Milnor. Dynamics in one complex variables: Introductory lectures. Preprint # 1990/5, SUNY Stony Brook, 1990.

[Mil92] J. Milnor. Remarks on iterated cubic maps. *Experimental Mathematics*, 1:5–24, 1992.

[Mil93] J. Milnor. Geometry and Dynamics of Quadratic Rational Maps. *Experimental Mathematics*, 2(1):37–83, 1993.

[Mil97] J. Milnor. On Rational Maps with Two Critical Points. Preprint ims97-10, SUNY Stony Brook, 1997.

[NF96] K. Nishizawa and M. Fujimura. Moduli spaces of maps with two critical points. *Special Issue No.1, Science Bulletin of Josai Univ.*, pages 99–113, 1997.

[NF99] K. Nishizawa and M. Fujimura. Bifurctions and Hyperbolic Components. In W. Takahashi et al., editors, *Proceedings of the 1998 International Conference on Nonlinear Analysis and Convex Analysis*, pages 289–296. World Scientific, 1999.

[NF00] K. Nishizawa and M. Fujimura. Chaotic bifurcations along Algebraic Curves. In S. Elaydi et al., editors, *Proceedings of the 4-th International Conference on Difference Equations: Communications in Diffence Equations*, pages 273–282. Gordon and Breach Sci. Pub., 2000.